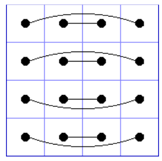
Theorem 8 [connection figure (1221 3443 5665 7887)]



(i) Let k,r,s,t,N be natural numbers (with k+s<r+1 and t<k+r+1) such that the 16 numbers

(*) 1, k+1, r+1, s+1, t+1, -k+r+1, k+r-t+1, -k+r-s+1,
N+k-r+s, N-k-r+t, N+k-r, N-t, N-s, N-r, N-k, N

are pairwise different and positive, then there exist (at least) 8 different general 4x4 magic squares M01,M02,...,M08 with entries from (*) and connection figure (1221 3443 5665 7887), namely:

	1	t+1	N-t	N
	N-s	N-k	k+1	s+1
M01 =	N+k-r+s	N-r	r+1	-k+r-s+1,
	-k+r+1	k+r-t+1	N-k-r+t	N+k-r

	N-k	N-s	s+1	k+1
	t+1	1	N	N-t
M02 =	k+r-t+1	-k+r+1	N+k-r	N-k-r+t,
	N-r	N+k-r+s	-k+r-s+1	r+1

M03,M04 are mirror images of M01,M02 from reflection at a vertical axis.
M05,M06,M07 and M08 are derived from M01,...,M04 by reflection at a horizontal axis.

(ii) Let k,r,s,t,N be natural numbers (with k<r+t+1, k+s<r+2t+1) such that the 16 numbers

(**) 1, k+1, r+1, s+1, t+1, k+r+1, -k+r+t+1, -k+r-s+2t+1,
N+k-r+s-2t, N+k-r-t, N-k-r, N-t, N-s, N-r, N-k, N

are pairwise different and positive, then there exist (at least) 8 different general 4x4 magic squares M09,M10,...,M16 with entries from (*) and connection figure (1221 3443 5665 7887), namely:

	t+1	1	N	N-t
	N-s	N-k	k+1	s+1
M09 =	N+k-r+s-2t	N-r	r+1	-k+r-s+2t+1,
	-k+r+t+1	k+r+1	N-k-r	N+k-r-t

	N-k	N-s	s+1	k+1
	1	t+1	N-t	N
M10 =	k+r+1	-k+r+t+1	N+k-r-t	N-k-r.
	N-r	N+k-r+s-2t	-k+r-s+2t+1	r+1

M11,M12 are mirror images of M09,M10 from reflection at a vertical axis.
M13,M14,M15 and M16 are derived from M09,...,M12 by reflection at a horizontal axis.

(iii) Let T be a symmetric subset of {1,...,N}, containing 1 as an element and let M be a general 4x4 magic square of connection figure (1221 3443 5665 7887) with entries from T.
If 1 is a diagonal element of M, then there exists a quadruple (k,r,s,t), such that T is the set (*) and M is one of the eight squares M01,...,M08.
If 1 does not belong to a diagonal of M, then there exists a quadruple (k,r,s,t), such that T consists of the elements (**) and M is one of the eight squares M09,...,M16.

(iv) Suppose (k,r,s,t) represents T as set (*). Call (k,r,s,t) “1-reduced”, when the inequalities 2*t+1 < k+r+1 < N and r < k+2*s hold.
Under the assumptions of (iii) with 1 as a diagonal element, there exists a unique 1-reduced (k,r,s,t).
Seven other quadruples represent T as set (*), too:
(N-r-1,N-k-1,s,N-k-r+t-1), (k,r,s,k+r-t), (N-r-1,N-k-1,s,N-t-1), (k,r,-k+r-s,t),
(N-r-1,N-k-1,-k+r-s,N-k-r+t-1), (k,r,-k+r-s,k+r-t), (N-r-1,N-k-1,-k+r-s,N-t-1).

Now suppose (k,r,s,t) represents T as a set (**). Call (k,r,s,t) “2-reduced”, if the inequalities k<r<k+2s-2t and 2t+r+1<N+k are valid.
Under the assumptions of (iii), with 1 not in any diagonal, there exists a unique 2-reduced (k,r,s,t), and seven other quadruples represent T as a set (**), too:
(r,k,-k+r-s+2t,-k+r+t), (r,k,N+k+r+s-2t-1,N+k-r-t-1), (r,k,N-s-1,N-t-1),
(k,r,-k+r-s+2t,t), (r,k,s,-k+r+t), (r,k,N-s-1,N+k-r-t-1), (r,k,N+k-r+s-2t-1,N-t-1).

Moreover, let z1 be the number of 1-reduced quadruples (k,r,s,t), such that T consists of the elements (*), and z2 be the number of 2-reduced quadruples (k,r,s,t), such that T is represented by the elements (**).
Then there are exactly 64*z1+64*z2 general 4x4 magic squares with entries from T.
For N<61 there are 114 possible values for (z1|z2), shown in the table below. An entry N in row z1 and column z2 means: N is the smallest value, that there exists a symmetric subset with 16 elements from {1,...,N}, containing 1, represented by z1 1-reduced quadruples as (*) and z2 2-reduced quadruples as set (**).

Proof

(i), (ii) can be verified easily. (iii) follows, when the involved linear equations for M are solved. (iv) results from 7 transformations for M01 and 7 transformations for M09, which let 1 and N fixed, they are shown in the appendix below. The values for (z1|z2) were found by computer experiment. For 31<N no new pair (z1|z2) was found.

Examples

(1) N=16, classical magic 4x4-squares:
There are z1=11 quadruples (k,r,s,t), which are 1-reduced, namely
(1,7,4,3), (1,11,8,3), (1,12,9,5), (1,13,8,5), (2,7,4,3), (2,9,4,1),
(2,11,8,3), (3,8,4,2), (3,10,6,2), (4,7,2,5), and (5,8,2,4).
There are z2=8 quadruples (k,r,s,t), which are 2-reduced:
(1,2,7,5), (1,4,7,3), (1,8,11,3), (2,4,7,3), (2,8,11,3), (3,8,5,1)),
(4,8,13,5), and (4,9,10,3).
Therefore there exist 64(z1+z2)=1216 classical magic 4x4-squares with connection figure (1221 3443 5665 7887).

(2) N=20, T={1,2,3,4,6,7,8,10,11,13,14,15,17,18,19,20}.
Here (z1|z2)=(1|0); (k,r,s,t)=(1,17,9,5). Therefore there are 64 general 4x4 magic squares of connection figure (1221 3443 5665 7887) with entries from T.

(3) The smallest N, which has a symmetric subset T of 16 elements, containing 1, with (z1|z2)=(0|0), is N=19, T={1,2,3,4,5,6,7,9,11,13,14,15,16,17,18,19}.
For N=16,17,18, and even N=20, every symmetric subset T, of {1,...,N} with |T|=16 and 1 as an element, allows general 4x4 magic squares with entries from T.

Table for (z1|z2)

		z2																			
		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
	0	19	20	20	22	20	20	22	25	25	29	29		29							
	1	20	20	18	18	18	18	22	24	25	31										
	2	22	22	20	20	21	20	22	23	25	31	21		21							
	3	20	22	20	20	21	22	23	23	26	19		21								
	4	25	25	22	20	24	23	21	24												
	5	27	21	25	21	27	23	20	27												
z1	6		25	23	25	23		19		24	28	26	22	25	24	25		28	25		22
	7						23		20		24	22		20	24				25		
	8					21				18		20	20	21	18	23		22			
	9							25		22	29	26	22	25	21						
	10								22		20	22	23	20	24						
	11								27	16		23	25	21	18						
	12									19	19	21									
	13																		17		

Appendix

```
                                                                   c01 c02 c03 c04
                                                                   c05 c06 c07 c08
7 mappings for M01, resp. M09, which let 1 and N fixed.  M01/M09 = c09 c10 c11 c12,
                                                                   c13 c14 c15 c16

         c01 c15 c14 c04     c01 c14 c15 c04     c01 c03 c02 c04
         c05 c11 c10 c08     c05 c06 c07 c08     c05 c11 c10 c08
M01 -->  c09 c07 c06 c12,    c09 c10 c11 c12,    c09 c07 c06 c12,
         c13 c03 c02 c16     c13 c02 c03 c16     c13 c15 c14 c16

         c01 c02 c03 c04     c01 c15 c14 c04     c01 c14 c15 c04     c01 c03 c02 c04
         c09 c06 c07 c12     c09 c11 c10 c12     c09 c06 c07 c12     c09 c11 c10 c12
         c05 c10 c11 c08,    c05 c07 c06 c08,    c05 c10 c11 c08,    c05 c07 c06 c08.
         c13 c14 c15 c16     c13 c03 c02 c16     c13 c02 c03 c16     c13 c15 c14 c16


         c13 c02 c03 c16     c16 c02 c03 c13     c04 c02 c03 c01
         c09 c10 c11 c12     c12 c10 c11 c09     c08 c10 c11 c05
M09 -->  c05 c06 c07 c08,    c08 c06 c07 c05,    c12 c06 c07 c09,
         c01 c14 c15 c04     c04 c14 c15 c01     c16 c14 c15 c13

         c01 c02 c03 c04     c13 c02 c03 c16     c16 c02 c03 c13     c04 c02 c03 c01
         c09 c06 c07 c12     c05 c10 c11 c08     c08 c10 c11 c05     c12 c10 c11 c09
         c05 c10 c11 c08,    c09 c06 c07 c12,    c12 c06 c07 c09,    c08 c06 c07 c05.
         c13 c14 c15 c16     c01 c14 c15 c04     c04 c14 c15 c01     c16 c14 c15 c13
```