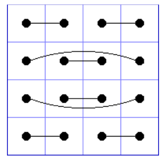
Theorem 2 [connection figure (1122 3443 5665 7788)]



(i) Let k,r,s,N be natural numbers, such that the 16 numbers

(*) 1,k,r,s,k+r-1,2r-1,s-k+1,k+2r-2,
N+1-s,N+1-r,N+1-k,N+k-s,N+2-k-r,N+3-k-2r,N+2-2r,N

are pairwise different and positive.
Then (*) is a symmetric subset of {1,...,N}, and there are (at least) 16 general 4x4 magic squares M01,...,M16 with entries from (*) and connection figure (1122 3443 5665 7788), namely:

$$M01 = \begin{array}{llll} 1 & N & 2r-1 & N+2-2r \\ s & N+2-k-r & k+r-1 & N+1-s \\ N+k-s & r & N+1-r & s-k+1 \\ N+1-k & k & N+3-k-2r & k+2r-2 \end{array},$$

M02 is the complement of M01 (each entry x is replaced by N+1-x), M03,M04 are obtained from M01,M02 by reflection at a vertical axis, M05,M06,M07, and M08 are the mirror images of M01,M02,M03, and M04 by reflection at a horizontal axis, and finally M09,...,M16 come from M01,...,M08 by exchange of the first entries of the second and third row and simultaneously the fourth entries of the second and third row of M01,...,M08.

(ii) Let k,r,s,N be natural numbers such that the 16 numbers

(**) 1,s,r,k,r-s+1,2r-k,k-s+1,2r-k-s+1
N+1-r,N+1-s,N+1-k,N-r+s,N-k+s,N+k-2r+1,N+k-2r+s,N

are pairwise different and positive.
Then (*) is a symmetric subset of {1,...,N} and there are (at least) 16 general 4x4 magic squares M17,...,M32 with entries from (**) and connection figure (1122 3443 5665 7788), namely:

$$M17 = \begin{array}{llll} k & N+1-k & 2r-k & N+k-2r+1 \\ 1 & N-r+s & r-s+1 & N \\ N+1-s & r & N+1-r & s \\ N-k+s & k-s+1 & N+k-2r+s & 2r-k-s+1 \end{array},$$

M18 is the complement of M17 (each entry x is replaced by N+1-x), M19,M20 are obtained from M17,M18 by reflection at a vertical axis, M21,M22,M23, and M24 are the mirror images of M17,M18,M19, and M20 by reflection at a horizontal axis, and finally M25,...,M32 come from M17,...,M24 by exchange of the first entries of the second and third row and simultaneously the fourth entries in the second and third row of M17,...,M24.

(iii) Let T be a symmetric subset of {1,...,N}, containing the element 1 and let M be a general 4x4 magic square with entries from T and connection figure (1122 3443 5665 7788).
If 1 is an entry of the first or fourth row of M, then there exists a triple (k,r,s) such that T is of the form (*) and M is one of the 16 squares M01,...,M16.
If 1 is the first or fourth entry of the second or third row of M, then there exits a triple (k,r,s) with: (**) are the entries of T, and M is one of the 16 squares M17,...,M32.
The second and the third entry in the second or third row of M cannot be 1.

(iv) Under the assumptions of (iii) with 1 as a diagonal element, there exists a unique triple (k,r,s) with 2s<N+k such that (k,r,s) and (k,r,N+k-s) both represent T as set (*).
Under the assumptions of (iii), with 1 not in any diagonal, there exists a unique triple (k,r,s) with k<r and 2r<N+s such that (k,r,s),(N-k+s,N-r+s,s),(N+k-2r+s,N-r+s,s), and (-k+2r,r,s) each represent T as set (**).
Moreover, let z1 be the number of triples (k,r,s), with 2s<N+k, such that T consists of the elements (*), and z2 be the number of triples (k,r,s), with k<r and 2r<N+s, such that T is represented by the elements (**).
Then there are exactly 16*z1+16*z2 general 4x4 magic squares with entries from T.
If N<61, then the following 26 pairs (z1,z2) are possible:
(0,0),(1,0),(0,2),(1,1),(2,0),(1,2),(2,1),(3,0),(2,2),(3,1),(4,0),
(2,3),(3,2),(4,1),(2,4),(3,3),(4,2),(4,3),(4,4),(4,5),(5,4),
(5,5),(6,4),(6,5),(8,5), and (8,6).

Proof

(i) and (ii) can be verified easily.
Using only linear algebra, the first part of (iii) is shown by solving the linear equations involved.
(iv),case (*) results from a transformation of M, which lets 1 and N fixed:
exchange of the first entries in rows 2 and 3 and, simultaneously, exchange of the fourth entries of the same rows.
(iv),case (**) results from 3 non-identical transformations of M, which let 1 and N fixed, described in the appendix below.
The possible pairs (z1,z2) for $N<61$ were found by computer experiment.
For $23<N$ no new pair (z1,z2) was found.

Examples

(1) $N=16$, classical 4x4 magic squares of connection figure (1122 3443 5665 7788):

There are $z1=8$ triples (k,r,s) with $2s<N+k$, representing {1,...,16} as set (*):
(2,3,8), (2,5,4), (2,7,6), (3,5,4), (5,2,8), (9,2,12), (9,3,10), and (9,4,11).

There are $z2=6$ triples (k,r,s) with $k<r$ and $2r<N+s$, representing {1,...,16} as set (**), namely
(4,6,2), (4,8,2), (4,8,3), (6,7,5), (10,11,9), and (10,12,9)
Therefore there are $16*8+16*6=224$ classical magic squares of the mentioned connection figure.

(2) $N=28$, T={1,2,3,4,5,6,11,12,17,18,23,24,25,26,27,28}:

There is $z1=1$ triple representing T as set (*), namely (2,3,12), and $z2=0$.
Therefore 16 general 4x4 magic squares of connection figure (1122 3443 5665 7788) can be built from set T.

(k,r,s)=(2,3,12): $1, k=2, r=3, s=12, k+r-1=4, 2r-1=5, s-k+1=11, k+2r-2=6,$
$N+1-s=17, N+1-r=26, N+1-k=27, N+k-s=18, N+2-k-r=25,$
$N+3-k-2r=23, N+2-2r=24, N=28.$

Appendix

3 additional transformations of M, case (**)

$$M \longrightarrow \begin{matrix} N-k+s & k-s+1 & N+k-2r+s & -k+2r-s+1 \\ 1 & r & N+1-r & N \\ N+1-s & N-r+s & r-s+1 & s \\ k & N+1-k & -k+2r & N+k-2r+1 \end{matrix}\ ,\quad M \longrightarrow \begin{matrix} N+k-2r+s & -k+2r-s+1 & N-k+s & k-s+1 \\ 1 & r & N+1-r & N \\ N+1-s & N-r+s & r-s+1 & s \\ -k+2r & N+k-2r+1 & k & N+1-k \end{matrix}\ ,$$

$$M \longrightarrow \begin{matrix} -k+2r & N+k-2r+1 & k & N+1-k \\ 1 & N-r+s & r-s+1 & N \\ N+1-s & r & N+1-r & s \\ N+k-2r+s & -k+2r-s+1 & N-k+s & k-s+1 \end{matrix}\ .$$