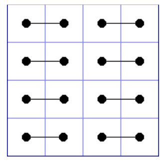
<u>Theorem 1.1</u> [connection figure (1122 3344 5566 7788)]



(i) Let k,r,s,N be natural numbers, such that the 16 numbers

(#) 1, k+1, r+1, s+1, k+r+1, k+s+1, r+s+1, k+r+s+1,
N-k-r-s, N-r-s, N-k-s, N-k-r, N-s, N-r, N-k, N

are pairwise different and positive.
Then there exist 384 general 4x4 magic squares M001,M002,...,M384 with entries from (#), of connection figure (1122 3344 5566 7788) namely:

M001 =

1	N	k+r+1	N-k-r
k+s+1	N-k-s	r+s+1	N-r-s
N-k	k+1	N-r	r+1
N-s	s+1	N-k-r-s	k+r+s+1

,

M002 =

1	N	N-k-s	k+s+1
N-k-r	k+r+1	r+s+1	N-r-s
k+r+s+1	N-k-r-s	N-r	r+1
N-s	s+1	k+1	N-k

,

M003 =

1	N	N-r-s	r+s+1
k+s+1	N-k-s	N-k-r	k+r+1
N-k	k+1	k+r+s+1	N-k-r-s
N-s	s+1	r+1	N-r

,

M004 =

1	N	k+r+1	N-k-r
N-r-s	r+s+1	N-k-s	k+s+1
N-k	k+1	N-r	r+1
k+r+s+1	N-k-r-s	s+1	N-s

,

M004 till M008 are derived from M001 till M004 by simultaneous exchange of their rows 1 with 2,the rows 3 with 4, and then the columns 1 with 2 and the columns 3 with 4. M009, M010,...,M016 arise from M001,...,M008 by reflection at a horizontal axis, further the squares M017 till M032 are the mirror images of M001,...,M016 from reflection at a vertical axis. M033,M034,...,M192 come from M001,...,M032 by application of the 5 non-identical permutations of three objects to the triple (k,r,s) and finally, M193,M194,...,M384 are derived from M001,...,M192 by replacement of each entry x by N+1-x (which has the same effect as simultaneous exchange of column 1 with column 2 and column 3 with column 4).

(ii) Let T be a symmetric subset of {1,...,N} with 16 elements, containing 1 as an element and let M be a general 4x4 magic square of connection figure (1122 3344 5566 7788) with entries from T. Then there exists a triple (k,r,s) of natural numbers with k<r<s such that T consists of the numbers (#). When 1 is a diagonal element of M, then M is one of the squares M001,...,M192; otherwise M is one of the squares M193,...,M384.

<u>Proof</u>

(i) can be verified easily. (ii) can be shown by solving the involved linear equations for M.

<u>Definition 1</u>

Let x1,x2,x3,x4 be natural numbers, such that x1<x2<x3<x4 and 1+x1+x2+x3+x4=N
and the 16 numbers

(*) S(1)=1,S(2)=1+x1,S(3)=1+x2,S(4)=1+x1+x2,
S(5)=1+x3,s(6)=1+x1+x3,S(7)=1+x2+x3,s(8)=1+x1+x2+x3,
S(9)=1+x4,S(10)=1+x1+x4,S(11)=1+x2+x4,S(12)=1+x1+x2+x4,
S(13)=1+x3+x4,S(14)=1+x1+x3+x4,S(15)=1+x2+x3+x4,S(16)=N

are pairwise different.
Call a set of 4 numbers {S(i1),S(i2),S(i3),S(i4)} with S(i1)+S(i2)+S(i3)+S(i4)=2(N+1) "correct", if 1,x1,x2,x3, and x4 each occur exactly 2 times, when the 4 numbers are represented by 1,x1,x2,x3,x4.

<u>Theorem 1.2</u>

Let x1,x2,x3,x4 be natural numbers, such that x1<x2<x3<x4 and 1+x1+x2+x3+x4=N
and the 16 numbers

(*) S(1)=1,S(2)=1+x1,S(3)=1+x2,S(4)=1+x1+x2,
S(5)=1+x3,s(6)=1+x1+x3,S(7)=1+x2+x3,s(8)=1+x1+x2+x3,
S(9)=1+x4,S(10)=1+x1+x4,S(11)=1+x2+x4,S(12)=1+x1+x2+x4,
S(13)=1+x3+x4,S(14)=1+x1+x3+x4,S(15)=1+x2+x3+x4,S(16)=N

are pairwise different.

Then the Set S of numbers (*) is a symmetric subset of {1,...,N}.

Let M=(cij), i,j =1,..,4 be a classical magic 4x4 square with entries from {1,...,16}
and correct rows, columns and diagonals (in the classical sense: every cij-1 is uniquely represented as sum of 1,2,4,8; sufficient: the main diagonal is correct).

The map k -> S(k) generates a general magic square M* = (S(cij)), i,j =1,...,4 with magic sum 2(N+1)

Example

N=22, x1=3, x2=5, x3=6, x4=7, S={1,4,6,7,8,9,10,11,12,13,14,15,16,17,19,22}
Take Duerer's magic square for M

```
   16 03 02 13       22 06 04 14
   05 10 11 08       07 11 13 15
M: 09 06 07 12   M*: 08 10 12 16
   04 15 14 01       09 19 17 01
```

Proof

Consider any entry cij of M. Then cij-1 can be uniquely written as a binary
0000,0001,0010,0011,0100,0101,0110,0111,1000,1001,1010,1011,1100,1101,1110,1111.
In M* read the above binaries as sums of x1,x2,x3,x4; f.i. read 0101 as x1+x3.
Then, because of correctness in any row, column, or diagonal the binary sum of entries cij-1 is 1111+1111=30,
the sum of entries S(cij)-1 in the corresponding row column or diagonal of M* is x4+x3+x2+x4+x3+x2+x1=2N-2, and therefore, every row, column or diagonal in M* sums to 2(N+1).

Theorem 1.3

Let N be a natural number and let S={s1,s2,...,s16} be a subset of {1,...,N} with 16 elements 1=s1<s2<s3<...<s15<s16=N.
Suppose that there exists a general 4x4-magic square with different entries from S, which belongs to the connection figure (1122 3344 5566 7788).
Then there exists a unique set of 4 numbers x1,x2,x3,x4 with x1<x2<x3<x4 and 1+x1+x2+x3+x4=N, such that the 16 numbers (*) of Theorem 1.1 are exactly the members of S.
x1,x2,x3,x4 can be found as follows:
x1=s2-1, x2=s3-1, if 1+x1+x2<s(5) then x3=s5-1 else x3=s4-1, and x4=s15-x2-x3-1.

Proof

This follows from (ii) of Theorem 1.1, with x1=k,x2=r,x3=s, and x4=N-k-r-s-1.

Remark 1

Every general 4x4 magic square with entry 1 and connection figure (1122 3344 5566 7788) can be generated via (1,2,4,8) --> (k,r,s,t), N=k+r+s+t+1, and the 384 mappings described in (ii) of Theorem 1.1, from only the one classical 4x4-magic square

```
 1  16   4  13
 6  11   7  10
15   2  14   3
12   5   9   8.
```

Remark 2

There is an imbedding map i from the set of general 4x4 magic squares of connection figure (1122 3344 5566 7788) into the set of general 4x4 magic squares of connection figure (1221 3443 5665 7887), namely:

```
    c01 c02 c03 c04        c01 c05 c06 c02
    c05 c06 c07 c08        c11 c13 c14 c12
i:  c09 c10 c11 c12  -->   c09 c15 c16 c10.
    c13 c14 c15 c16        c03 c07 c08 c04
```

This is a consequence of four equations, valid for every general 4x4 magic square of connection figure (1122 3344 5566 7788):

c01+c03+c09+c11=2(N+1), c02+c04+c10+c12=2(N+1), c05+c07+c13+c15=2(N+1), c06+c08+c14+c16=2(N+1).